Lecture 1. §1. 黎曼联络

1. 曲面片 (Gauss 绝妙定理)

$\vec{r}: U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\qquad \underline{e^3}$

$(u^1, u^2) \longmapsto \vec{r}(u^1, u^2)$

$\vec{r}_i \cdot \vec{r}_j = g_{ij} \qquad g = g_{ij}\, du^i du^j \left(= \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}\, du^i du^j \right)$ 第一基本形式

$\vec{n} = \dfrac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}, \qquad \begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \end{pmatrix} = -\begin{pmatrix} w_1^1 & w_1^2 \\ w_2^1 & w_2^2 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}, \qquad w = \Omega g^{-1}$

可对角化，记为 w.

Ω: 第二基本形式系数矩阵.

w 有两个特征值. k_1, k_2. (与 n 取法有关)

Gauss 曲率 $K \triangleq k_1 k_2$. $\left(\begin{array}{l} \text{Gauss 绝妙定理} \\ K \text{ 与 } \vec{n} \text{ 无关!!!} \\ \text{只与“I”有关} \end{array} \right)$

$\Downarrow$

$K = |W| = \dfrac{|\Omega|}{|g|}$

$r_{ij} = \Gamma_{ij}^1 r_1 + \Gamma_{ij}^2 r_2 + \Omega_{ij} n$

("e^2" $\quad r_{ij} = r_{ji}$) $\Rightarrow \Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{il,j} + g_{lj,i} - g_{ij,l})$

求偏导 $\frac{\partial g_{il}}{\partial x^j}$

$r_{ijk} = (\Gamma_{ij}^l r_l + \Omega_{ij} n)_k$

$= \left(\Gamma_{ij,k}^l + \Gamma_{ij}^m \Gamma_{mk}^l - \Omega_{ij} w_k^l \right) r_l + (?+??)\, n$

$\Omega_{ij} w_k^l = \Omega_{ij} \Omega_{km} g^{ml}$

("C^3" $\quad \because r_{ijk} = r_{ikj}$)

$\therefore \underbrace{\Gamma_{ij,k}^l - \Gamma_{ik,j}^l + \Gamma_{ij}^m \Gamma_{mk}^l - \Gamma_{ik}^m \Gamma_{mj}^l}_{= R_{ijk}^l} = (\Omega_{ij} \Omega_{km} - \Omega_{ik} \Omega_{jm}) g^{ml}$

再记 $R_{imjk} \triangleq R_{ijk}^l \cdot g_{lm}$

$= \left(\Gamma_{ij,k}^l - \Gamma_{ik,j}^l + \Gamma_{ij}^m \Gamma_{mk}^l - \Gamma_{ik}^m \Gamma_{mj}^l \right) g_{lm} = \Omega_{ij} \Omega_{km} - \Omega_{ik} \Omega_{jm}$

$K = \dfrac{R_{1212}}{|g|}$

$$(r_{ij})^T = \Gamma_{ij}^1 r_1 + \Gamma_{ij}^2 r_2$$

切平面投影

$$\therefore \; ((r_{ij}^T)_k)^T = \Gamma_{ij,k}^{l} + \Gamma_{ij}^{m}\Gamma_{mk}^{l} - \Omega_{ij}\Omega_{km}g^{ml}$$

$$\Rightarrow \langle ((r_{ij}^T)_k)^T - ((r_{ik}^T)_j)^T, r_m \rangle = R_{imjk}$$

$$\begin{cases} (r_{ij})^T = (r_{ji})^T \Rightarrow \Gamma_{ij}^k = \Gamma_{ji}^k \\ \langle r_i, r_j \rangle_k = \langle r_{ik}^T, r_j \rangle + \langle r_i, r_{jk}^T \rangle \end{cases}$$

切向量场

$$X(u^1, u^2) = X^i r_i \quad , \; Y = Y^j r_j$$

方向导数: $\lim\limits_{t\to 0} \dfrac{f(p+tx) - f(p)}{t}$

$\mathbb{R}^3$. 方向导数: $\widetilde{D}_X Y = X^i\left(Y_i^j r_j + Y^l r_{li}\right)$

Y沿X方向

记: $D_X Y = X^i\left(Y_i^j r_j + Y^l r_{li}^T\right)$

Recall.

$$\begin{cases} (r_{ij})^T = (r_{ji})^T \iff D_X Y - D_Y X = [X, Y] \\ \langle r_i, r_j \rangle_k = \langle r_{ik}^T, r_j \rangle + \langle r_i, r_{jk}^T \rangle \end{cases}$$

$$\iff X\langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$$

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2. 微分流形

[WSY], [CC], [Pe]

在 M^n 拓扑流形上赋予 C^∞-微分结构

$$\{(U, \varphi_U), (V, \varphi_V), \cdots\} \quad ([CC])$$

与 $\mathbb{R}^n$ 开集同胚 $\qquad \varphi_U \circ \varphi_V^{-1}, \varphi_V \circ \varphi_U^{-1}$ 光滑 (相容性)

$$f \in C^\infty(M) \iff f|_U \in C^\infty(U)$$

$\hookrightarrow (u^1, \cdots, u^n)$

切向量场: $X \in \underbrace{\Gamma(TM)}_{\text{切丛截面}} \iff X|_U = X^i \frac{\partial}{\partial u^i}$

限制在坐标卡上.

$$X(f)|_U = X^i \frac{\partial f}{\partial u^i}$$

余切 … $\omega \in \Gamma(T^*M) \iff \omega|_U = f_i \, du^i$

并非内积. 而是配对.

$$\langle X, \omega \rangle|_U = (X^1, \cdots, X^n) \begin{pmatrix} \frac{\partial}{\partial u^1} \\ \vdots \\ \frac{\partial}{\partial u^n} \end{pmatrix} (du^1, \cdots, du^n) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$= X^i f_i$$

1. $X(af) = aX(f)$, $X(af+bg) = aX(a) + bX(g)$.

2. $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$

定义: $[X, Y] \triangleq XY - YX \in \Gamma(TM)$ (check!)

"T_r^s 型张量丛"

"双重线性函数全体."

$V^* = \{e_i^*\}$, $\mathcal{L}(V \times V) = V^* \otimes V^*$, $e_i^* \otimes e_j^*(v_1, v_2)$

$\qquad\qquad\qquad\| \qquad\qquad\qquad\qquad\qquad = e_i^*(v_1) \cdot e_j^*(v_2)$

$\qquad\qquad\{e_i^* \otimes e_j^*\}$

$$\underset{\in M}{T_r^s(p)} = \underbrace{T_p \otimes \cdots \otimes T_p}_{r\text{个}} \otimes \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{s\text{个}}.$$

(r, s) 型张量场 $\in \Gamma(T_r^s(M))$

例: 黎曼度量 (M上的). g

(存在性: 单位分解定理保证.)

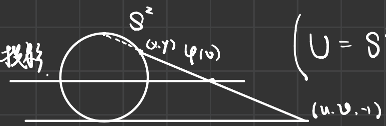
$$\langle f_1 X, f_2 Y \rangle = f_1 f_2 (X^1, X^2)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$$

g 是 M 上处处正定的 $(0,2)$-张量场

$$\Longleftrightarrow \quad g\big|_U = (du^1, \cdots, du^n)\underbrace{(g_{ij})}_{\text{对称正定}}\begin{pmatrix} du^1 \\ \vdots \\ du^n \end{pmatrix} = g_{ij}\, du^i \otimes du^j$$

例: $S^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$.

球极投影.



打到南极点切平面

$$\left(U = S^2 \setminus \{\text{北极点}\},\ \varphi_U \right)$$

$$\varphi_V^{-1} = \left(\frac{u}{\frac14(u^2+v^2)+1},\ \frac{v}{\sim},\ \frac{\frac14(\sim)-1}{\frac14(\sim)+1} \right)$$

作业: ① 计算 $\left(V = S^2 \setminus \{\text{南极点}\},\ \varphi_V \right)$

② 证明: $\varphi_U \circ \varphi_V^{-1}$, $\varphi_V \circ \varphi_U^{-1}$ 光滑.

③ 证明: $g_U = g_V$ (在 $U \cap V$ 上). 第一基本形式 (u^1, u^2), (v^1, v^2) 一个表示另一个.

④ 求 K.

Leture 2.

$$g = g_{ij}\, du^i \otimes du^j$$

$$g(X, Y) = (X^1, \cdots, X^n)\begin{pmatrix} \frac{\partial}{\partial u^1} \\ \vdots \\ \frac{\partial}{\partial u^n} \end{pmatrix}(du^1, \cdots, du^n)\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$= (X^1, \cdots X^n)(g_{ij})\begin{pmatrix} Y^1 \\ \vdots \\ Y^n \end{pmatrix}$$

$$g_{ij} = g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) \qquad T_p M. \quad \text{or } M_p$$

例: $\mathbb{R}^n$. $\quad g_0 = (dx^1)^2 + \cdots + (dx^n)^2$

作业：$g_U = g_V$（在 $U\cap V$ 上）　$g_U = g_V = g_0|_{S^{n-1}}$　$dx^2+dy^2+dz^2$.

例：$N^m \hookrightarrow (M^n, g)$
子流形.

Check $g|_N$ 是 N 上一个黎曼度量

3. 黎曼联络

1. 仿射联络. 若 ∃ $D: \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$（向量场全体）

s.t. $\forall X, Y, Z \in \Gamma(T\mathbb{R}^3)$　M（微分流形）

(1) $D_{fX+hY} Z = f D_X Z + h D_Y Z$

(2) $D_X(fY) = X(f)Y + f D_X Y$

(3) $D_X(Y+Z) = D_X Y + D_X Z$

则称 D 为 M 上的一个仿射联络.

注：$D_X Y|_p$ 仅由 $\gamma(t)|_{[0,\varepsilon)} \in C^\infty$（其中 $\gamma(0)=p$, $\dot\gamma(0) = X|_p$）
上的 Y 决定.

Thm（黎曼几何基本定理）

$\boxed{\nabla_X Y}$ 在 (M,g) 上 ∃! 联络 D s.t. $\forall X, Y, Z \in \Gamma(TM)$，有

$$\begin{cases} X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle \\ D_X Y - D_Y X = [X, Y] \end{cases}$$

无挠性.

$\Updownarrow$ $r_{ij}^T = r_{ji}^T$

$\Updownarrow$ 在曲面片上（自然基）

$g_{ij,k} = \langle r_{ik}^T, r_j\rangle + \langle r_i, r_{jk}^T\rangle$

证明：（唯一性）

取 U. 令 $g|_U = g_{ij}\, du^i \otimes du^j$，$D_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \Gamma_{ij}^k \frac{\partial}{\partial u^k}$

只需证 Γ_{ij}^k 可由 g_{ij} 表示即可

由无挠性，有 $\Gamma_{ij}^k = \Gamma_{ji}^k$

由保持内积有. $g_{ij,k} = \Gamma_{ik}^l g_{lj} + \Gamma_{jk}^l g_{li}$，对 i, j, k 轮换. 并

对三式求和

$\therefore \Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{il,j} + g_{lj,i} - g_{ij,l})$. ✓

(存在性) [CC]

只需证明在 U, V 上的 D 是一样(相同)即可 :) □

作业: 1. 证存在性

2. 不用坐标表示 $\underline{\langle D_X Y, Z \rangle}$

唯一性 ↙ ↘ $D_Y X$, $[X, Y]$. 反复化三次.

3. 取 $N \hookrightarrow (M, g)$ (嵌入), $\forall X, Y \in \Gamma(TN)$

令 $\bar{D}_X Y = (D_X Y)^T$

证明: $\bar{D}$ 为 $(N, g|_N)$ 上的 L-C 联络.

(M, g), 在 U 上. $X = X^i \frac{\partial}{\partial u^i}$, $Y = Y^i \frac{\partial}{\partial u^i}$

$$D_X Y \big|_U = X^i \left(Y_i^j \frac{\partial}{\partial u^j} + Y^j \Gamma_{ji}^k \frac{\partial}{\partial u^k} \right)$$

$$= X^i (Y_i^k + Y^j \Gamma_{ji}^k) \frac{\partial}{\partial u^k}$$

Def: 给定 (M, g) 上一条 C^∞ 的 $\gamma(t)\big|_{[a,b]}$

若 $D_{\dot\gamma(t)} Y = 0$, 则称 Y 沿 $\gamma(t)$ 平行.

令 $Y(t) = Y^i(\gamma(t)) \frac{\partial}{\partial u^i}$, 则 $\quad X^i(t) (Y_i^k + Y^j \Gamma_{ji}^k) \frac{\partial}{\partial u^k} = 0$.

$\dot\gamma(t) = X^i(t) \frac{\partial}{\partial u^i}$

$$\Rightarrow X^i(t) (Y_i^k + Y^j \Gamma_{ji}^k) = 0$$

$k = 1, 2, \cdots, n$

Def: 若 $D_{\dot\gamma} \dot\gamma = 0$ 则称 $\gamma(t)\big|_{[a,b]}$ 为一条测地线 (自平行曲线).

令 $\gamma(t) = (u^1(t), \cdots, u^n(t))$

$$\text{则} \quad \frac{d^2 u^k}{dt^2} + \frac{du^i}{dt} \frac{du^j}{dt} \underline{\Gamma_{ji}^k (u^1(t), \cdots, u^n(t))} = 0. \qquad k = 1, 2, \cdots, n.$$

光滑的

Thm：给定 (M,g)，$p\in M$，$v\in T_pM$，则 $\exists!$ 测地线 $\gamma(t)|_{[0,\varepsilon)}$，

$\gamma(0)$　$\dot\gamma(0)$

$\gamma(0)=p$，$\dot\gamma(0)=v$。

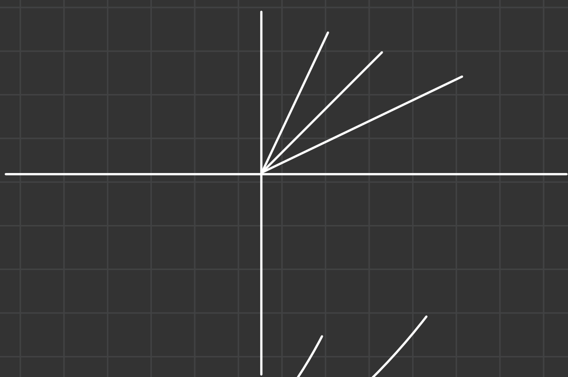
（由 O.D.E. 解存在唯一性）

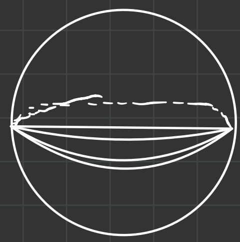
Remark：① 对于测地线 $\gamma(t)|_{[a,b]}$，$|\dot\gamma(t)|\equiv C$ $\left(\because \dot\gamma(t)\langle\dot\gamma(t),\dot\gamma(t)\rangle = 2\langle D_{\dot\gamma}\dot\gamma,\dot\gamma\rangle=0\right)$

② 过 p 点以 λv（$\lambda>0$）为初始切向的测地线为 $\tilde\gamma(t)=\gamma(\lambda t)$。

$\because$ $(\tilde\gamma'(t)=\lambda\gamma'(t))$ 作了个伸缩，换了个参数，仍为测地线 ①

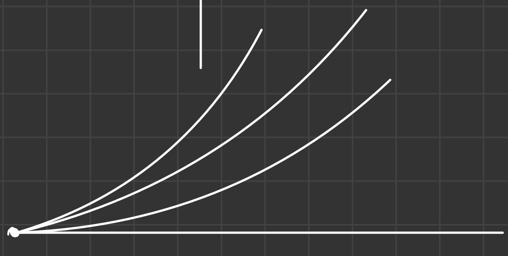
② 利用唯一性。





克莱因圆盘。

双曲。



"喇叭嘴"

出发后的走向由几何体形态决定。

$\Gamma_{ij}^{T} \longleftrightarrow D_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}$

(M, g), $\gamma(t) = (u^1(t), \cdots, u^n(t))$, $D_{\dot\gamma(t)} \dot\gamma(t) = 0$. $\dot\gamma(t) = \sum_i \dot u^i(t) \frac{\partial}{\partial u^i}$

$\dot\gamma(t)(f) = \frac{d f(\gamma(t))}{dt} = \sum \dot u^i \frac{\partial f}{\partial u^i}$. 计算得测地线方程.

给定 $p \in M$, $\exists\, \varepsilon > 0$ s.t. $\forall\, v \in T_pM$ 且 $|v| = 1$. $\exists !\ \gamma_v(t)\Big|_{t \in [0, \varepsilon)}$ 是测地线

且 $\gamma_v(0) = p$, $\dot\gamma_v(0) = v$

每个方向光滑依赖于 (t, v) 初值. 而初值切向构成一个 S^{n-1} (compact). ($\mathbb{R}^n$ 中考虑)

定义: $\exp_p : B_0(\varepsilon) \subset T_pM \longrightarrow M$

$t v \longmapsto \gamma_v(t)$ 弧长参数

$0 \le t < \varepsilon$. $|v| = 1$

性质: $d\exp_p\Big|_0 = id : T_pM \longrightarrow T_pM$.

进而在 ε 充分小时. $\exp_p\Big|_{B_0(\varepsilon)}$ 是微分同胚

证明: $\forall\, w \in T_pM$, 取 $\underline{\text{曲线}}$ tw $(t \ge 0)$, 要证: $\exp_p(tw)$ 在 $t=0$ 时切向为 w
切向为 w.

$\exp_p(tw) = \exp_p\left(t \frac{w}{|w|} \cdot |w|\right) = \gamma_{\frac{w}{|w|}}(t|w|)$, 其中 $t \in [0, \frac{1}{|w|}\varepsilon)$

从 exp 定义可知. :)

注: $\forall\, p$, $\exists\, U_p \subset M$ $\varepsilon > 0$ s.t.

(Lie Group). $\forall\, q \in U_p$. $\exp_q\Big|_{B_0(\varepsilon)}$ 都是微分同胚

性质. T_pM 的直角坐标系 $(x^1, \cdots, x^n)$ 可自然地看为 M 在 p 处的 (法) 坐标系,

$$\text{而且} \begin{cases} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle\Big|_p = \delta_{ij} \\ D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\Big|_p = 0 \quad (\Gamma_{ij}^k = 0) \end{cases}$$

证明: (*) $i=j$ ✓ ($\exp_p\big(\underset{(0,\cdots,0,x_i,0,\cdots,0)}{\downarrow}\big)$ 是测地线)

$i\neq j$, $\because \gamma(t)=\exp_p\big((0,\cdots,\overset{i}{t},\cdots,\overset{j}{t},\cdots,0)\big)$ 为测地线

而且 $\dot\gamma(t)=\frac{\partial}{\partial x^i}+\frac{\partial}{\partial x^j}$

$\therefore D_{\frac{\partial}{\partial x^i}+\frac{\partial}{\partial x^j}}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial x^j}\right)=0.$ linear expansion.

$2D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=0$ □

推论: $\left(g_{ij}(x^1,\cdots,x^n)\right)=I_n+O\left((x^1)^2+\cdots+(x^n)^2\right)$

($\because \Gamma_{ij}^k=0 \Rightarrow g'=0$. $\therefore$ Taylor展开无一阶项)

作业: 给定 $\exp:[0,l]\times[0,\varepsilon]\longrightarrow M$

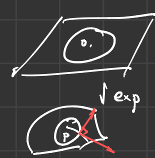
$(t,u)\longmapsto \exp t(X_0+uY)$

记 $U(t)=\frac{\partial}{\partial u}\exp t(X_0+uY)$

证明. $\dot U(t)\big|_{t=0}=Y$ ([WSY])

(Gauss lemma)

性质: p处的径向测地线垂直于 $\exp_p(\partial B_0(\rho))$, 其中 $\rho<\varepsilon$



证明: 取 T_pM 的极坐标 $(\rho,\theta^1,\cdots,\theta^{n-1})$ $\mathbb{R}\times S^{n-1}$

记 $\partial\rho\triangleq d\exp\left(\frac{\partial}{\partial\rho}\right)$

$\partial\theta^i\triangleq d\exp\left(\frac{\partial}{\partial\theta^i}\right)$

要证: $\langle\partial\rho,\partial\theta^i\rangle=0$

$\because \partial\rho\langle\partial\rho,\partial\theta^i\rangle=\langle\underset{\overset{\|}{0}}{D_{\partial\rho}\partial\rho},\partial\theta^i\rangle+\langle\partial\rho,D_{\partial\rho}\partial\theta^i\rangle$

$=\langle\partial\rho,D_{\partial\theta^i}\partial\rho\rangle=\frac{1}{2}\partial\theta^i\underset{\overset{\|}{1}}{\underline{\langle\partial\rho,\partial\rho\rangle}}$

$=0$

ρ: 测地线

$\therefore$ 只要说明 $\lim\limits_{\rho\to0}\langle\partial\rho,\partial\theta^i\rangle=0.$ ✓

$\lim\limits_{\rho\to0}|\partial\theta^i|=0.$

e.g. $\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta.\end{cases}$ $\partial\theta^i=\frac{\partial x^1}{\partial\theta^i}\frac{\partial}{\partial x^1}+\frac{\partial x^2}{\partial\theta^i}\frac{\partial}{\partial x^2}$

法坐标

推论：(M,g) 在极坐标系 $(\rho,\theta^1,\cdots,\theta^{n-1})$ 下，

$$\left(g_{ij}(\rho,\theta^1,\cdots,\theta^{n-1})\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

其中 $* = \rho^2\left(I_{n-1} + O(\rho^2)\right)$

性质：给定 (M,g)，$p\in M$，设 $\exp_p|_{B_0(\varepsilon)}$ 是微分同胚

则 $\forall q \in \exp(B_0(\varepsilon))$，则在 $\exp(B_0(\varepsilon))$ 上连接 p,q 间

的曲线中，测地线最短

证明：$\forall$ C^∞ 弧长参数化曲线 $C:[0,l] \longrightarrow \exp(B_0(\varepsilon))$

要证 $l(C) \geq l(\underbrace{[p,q]}_{\text{测地}})$

设 C 在 $(\rho,\theta^1,\cdots)$ 下为 $(\rho(s),\theta^1(s),\cdots)$

$\therefore l(C) = \int_0^l \sqrt{(\dot\rho(s))^2 + \text{非负项}}\, ds \geq \int_0^l \sqrt{(\dot\rho(s))^2}\, ds \geq \int_0^l \dot\rho(s)\, ds = \rho(q)$
$= l([p,q])$

$C' = \dot\rho(s)\frac{\partial}{\partial\rho} + \dot\theta^1(s)\frac{\partial}{\partial\theta^1}\cdots$

(M,d) $d(x,y) = \inf\limits_{\substack{\gamma\text{是连 }x,y\text{ 的}\\ \text{可求长曲线}}} (l(\gamma))$

定理 (Hopf-Rinow)

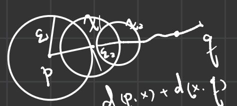
设 (M,g) 完备，则 M 上任意两点间 $\exists$ 最短线。
(距离拓扑完备)

$\Longleftrightarrow$ 从一点出发可无限延伸

$\Longleftrightarrow$ 有界闭子集一定紧

($\Longleftrightarrow$ M 上任意测地线可无限延伸)

证明：



$x \in \partial B_p(\varepsilon)$

§2. 曲率张量

一. $R_{imjk} = \langle (r_{ij}^T)_k^T - (r_{ik}^T)_j^T, r_m \rangle$

$$K = \frac{R_{1212}}{|g|}$$

(M, g) U

$$R_{imjk} := \langle D_{\frac{\partial}{\partial u^k}} D_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i} - D_{\frac{\partial}{\partial u^j}} D_{\frac{\partial}{\partial u^k}} \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^m} \rangle$$

定义：$R(X, Y)Z = -D_X D_Y Z + D_Y D_X Z + D_{[X,Y]} Z$

作业：$R(X,Y)Z$ 关于 3 个变量均函数线性 ($\Rightarrow$ 定义了一个 $(1,3)$ 张量场)

进而可定义 $R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$
(R 是一个 $(0,4)$ 型张量场)

作业：在 U 上，用 g_{ij} 的一、二阶导 ($\Gamma_{ij}^k \cdots$) 来表示

$$R_{ijkl} \triangleq R(\partial u^i, \partial u^j, \partial u^k, \partial u^l)$$

"1-1 加保距"

注(作业) $R(X, Y, Z, W)$ 在等距下不变 (i.e. 若 $\varphi: M \longrightarrow M'$ 是一个等距映射

则 $R'(X', Y', Z', W') = R(X, Y, Z, W)$.

$D'_{X'} Y' = d\varphi(D_X Y)$ $\quad$ $X' = d\varphi(X) \cdots$)

性质：(1) $R(X, Y)Z = -R(Y, X)Z$ ✓

(2) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

第一 Bianchi 恒等式

(3) $R(X, Y, Z, W) = -R(X, Y, W, Z)$

(4) $R(X, Y, Z, W) = R(Z, W, X, Y)$

证明：(2) ✓ 不妨 $[X, Y] = 0$.

由注，只需对 X, Y, Z, W 为自然基时证明即可

(3) 只需证 $R(X, Y, Z, Z) = 0$ 即可.

$$\| \quad \langle -D_X D_Y Z + D_Y D_X Z, Z \rangle$$

$$\| \quad -X\langle D_Y Z, Z \rangle + \cancel{\langle D_Y Z, D_X Z \rangle} + Y \langle D_X Z, Z \rangle - \cancel{\langle D_X Z, D_Y Z \rangle}$$

$$= -\frac{1}{2} XY\langle Z,Z\rangle + \frac{1}{2} YX\langle Z,Z\rangle$$
$$= 0$$

(4) $R(Z,W,X,Y) = -R(W,X,Z,Y) - R(X,Z,W,Y)$
$$= \cdots$$
$$= 2R(X,Y,Z,W) - R(Z,W,X,Y).$$

第二 Bianchi 恒等式

$$(D_X R)(Y,Z)W + (D_Y R)(Z,X)W + (D_Z R)(X,Y)W = 0$$

$$\| D$$

$$D_X(R(Y,Z)W) - R(D_X Y, Z)W - R(Y, D_X Z)W - R(Y,Z)D_X W$$

设 K 为 (r,s) 型张量场，定义 DK 为一个 $(r,s+1)$ 型张量场

满足 $DK(\omega_1,\cdots\omega_r, X_1,\cdots,X_s, Y)$

$$= (D_Y K)(\omega_1,\cdots,\omega_r, X_1,\cdots,X_s)$$
$$= D_Y\left(K(\omega_1,\cdots\omega_r, X_1,\cdots X_s)\right) - K(D_Y\omega_1,\omega_2,\cdots) - \cdots$$
$$- K(\omega_1,\cdots X_{s-1}, D_Y X_s)$$

如 $(D_Y\omega_1)(X) = Y(\omega_1(X)) - \omega_1(D_Y X)$

尤其是 $\omega = df$ $(Df(X) = X(f))$ if $\underbrace{DDf}_{D^2 f}(X,Y) = (D_Y Df)(X)$
$$= Y(X(f)) - (D_Y X)f$$

（当 $M = \mathbb{R}^n$，且 $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ 时 $DDf(X,Y) = \frac{\partial^2 f}{\partial x^i \partial x^j}$）

$\therefore D^2 f$ 为 f 的 Hessian，

定义：$\Delta f\big|_p = \mathrm{trace}(D^2 f) = \sum_{i=1}^n D^2 f(e_i, e_i)$，其中 $e_1,\cdots,e_n$ 为 T_pM 的一组单位正交基.

作业：(1) $D^2 f(X,Y) = D^2 f(Y,X)$

(2) $D^2 f$ 是一个 $(0,2)$ 型张量.

(3) 证明 Δf 定义的合理性（与坐标选取无关）.

作业：在 (M,g) U 上，证明

(1) $\Delta f = g^{ij} D^2 f\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$

(2) $\Delta f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k}\left(g^{ik}\sqrt{|g|}\frac{\partial f}{\partial x^i}\right)$

三、Sec_M，Ric_M，S_M

1. 给定 T_pM 中一个 2 维子空间 Π，取其一组基 v_1, v_2.

定义 $K(\Pi) \triangleq \dfrac{R(v_1, v_2, v_1, v_2)}{|v_1 \wedge v_2|^2}$　　注：$K(\Pi)$ 是等距的不变量

作业. 证明定义的合理性.

$Sec_M \geq K$　即一点所有截面的曲率 $\geq K$

合理

$Ric(X, Y) \triangleq \sum_{i=1}^{n} R(e_i, X, e_j, Y)$，其中 $e_1, \cdots, e_n$ 是 T_pM 单位正交基

给定 T_pM 中一个单位切向 X，定义 $Ric(X) \triangleq Ric(X, X)$

$S(p) \triangleq \mathrm{trace}(Ric(X, Y))$

$Ric_{M^n} \geq (n-1)K$

$Sec \equiv k$

单连通，完备的流形唯一.

Rmk: Schur theorem.

M 有体对称性（如何描述）群作用

齐性空间

$R(X, Y, Z, W)$

"R" $= f_{i_1 i_2 i_3 i_4} \, du^{i_1} \otimes du^{i_2} \otimes du^{i_3} \otimes du^{i_4}$

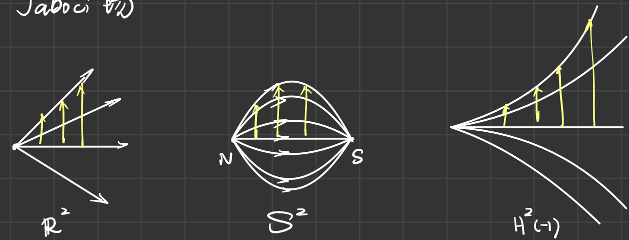
$R(v_1, v_2, v_1, v_2)$　　$v_j \in T_pM$

若 $X_1|_p = X_2|_p$，则 $R(X_1, Y, Z, W)|_p = R(X_2, Y', Z', W')|_p$

若 $X|_p = 0$，则 $R(X, Y, Z, W)|_p = 0$?

$\exists f, X_1$, s.t. $f X_1 = X$

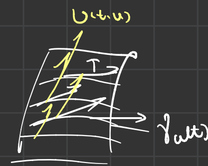
§3. Jaboci 场



取 C^∞ 的 $\gamma:\ [a,b]\times[a,b] \longrightarrow (M,g)$

$$(t,u) \longmapsto \gamma_u(t)$$

其中 $\gamma_u(t)$ 均为测地线.



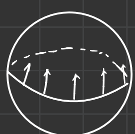
记 $T(t,u) = d\gamma(\frac{\partial}{\partial t})$

$U(t,u) = d\gamma(\frac{\partial}{\partial u})$ $\quad ([T,U]=0)$

$$D_T D_T U = D_T D_U T - D_U \underbrace{D_T T}_{=0} - D_{[T,U]} T = -R(T,U)T$$

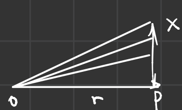
$$\therefore D_T D_T U + R(T,U)T = 0 \quad (*)$$

定义. 给定测地线 $\gamma(t)\big|_{[a,b]}$ 上的向量场 $U(t)$. 若 $U(t)$ 满足$(*)$ 则称 $U(t)$ 为沿 γ 的一个 Jacobi 场.

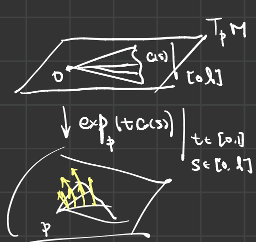
反例:  不是 Jacobi 场

例. $\exp_p \frac{t}{r}(p+uX)$ $\quad t\in[0,r],\ u\in[0,\varepsilon]$

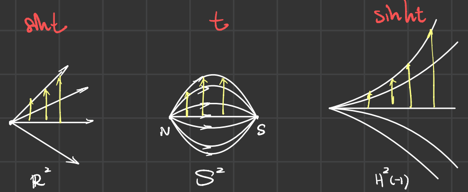
$$\begin{cases} U(0)=0 \\ U'(0)=\frac{1}{r}X \end{cases}$$



作业.



证明: $\begin{cases} U(0)=0 \\ U'(0,s)=\dot{c}(s) \end{cases}$



可令 $U(t)=f(t)\cdot E(t)$ 其中 $\dot{E}(t)=0$，$|E|=1$
$\dot{E}=D_T E$

$\ddot{U}+R(T,U)T=0$

可得 $\ddot{f}+f\cdot k=0$，其中 $f(0)=1$，$\dot{f}(0)=0$
k：Gauss 曲率

$$\sec\equiv k \Rightarrow g=\begin{cases} d\rho^2+\sin^2\rho\, d\theta^2 & k=1\\ d\rho^2+\rho^2 d\theta^2 & k=0\\ d\rho^2+\sinh^2\rho\, d\theta^2 & k=-1\end{cases}$$

$$g=\frac{1}{\left[1+\frac{k}{4}(x_1^2+\cdots+x_n^2)\right]^2}\left(dx_1^2+\cdots+dx_n^2\right)$$

可令 $\dot\gamma(t)=e_1(t)$ 且 $e_1(t),\cdots,e_n(t)$ 沿 $\gamma(t)$ 平行且单位正交.

设 $U(t)=f^i(t)\,e_i(t)$

$\therefore (*)$ 为 $\ddot{f}^i(t)\,e_i(t)+\underline{R(e_1,\ f^j(t)e_j(t))\,e_1}=0$

$$R(e_1,f^j(t)e_j(t))e_1=\sum_{i=1}^{n}\left(R(e_1,f^j(t)e_j(t))e_1,\ e_i\right)e_i$$

$\therefore\ \ddot{f}^i(t)+f^jR(e_1,e_j,e_1,e_i)=0$
$i=1,2,\cdots,n.$

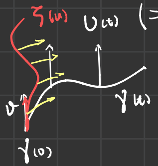
性质：给定 (M,g) 上一条测地线 $\gamma(t)\big|_{[0,l]}$ 及 $v,w\in T_pM$，其中 $p=\gamma(0)$.

则 (1) $\exists!$ 的 Jacobi 场 $U(t)$ s.t. $U(0)=v$. $\dot{U}(0)=w$

(2) $U(t)$ 沿 γ 零点离散 或 $U(t)\equiv0$.

性质：$U(t)$ 是沿 $\gamma(t)$（测地线）的 Jacobi 场 $\Leftrightarrow$ $U(t)$ 是关于 $\gamma(t)$ 的一个单参数测地线族的横截向量场在 $\gamma(t)$ 上的限制

证明：$(\Leftarrow)$ ✓



$(\Rightarrow)$ 设 $U(0)=v$. $\dot U(0)=w$

取沿 $\zeta(u)$ 平行的 $T(u)$, $W(u)$, 其中 $T(0)=\dot\gamma(0)$

$W(0)=w$

令 $\gamma_u(t) \triangleq \exp_{\zeta(u)} t\big(T(u)+uW(u)\big)$

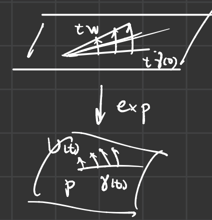
(当 $v=0$, $\gamma_u(t)=\exp_p t(\dot\gamma(0)+uw)$)

$$V(t)=\frac{\partial}{\partial u}\gamma_u(t)\Big|_{u=0}$$

[WST] → 只需证 $V(0)=v$, $\dot V(0)=w$ 即可 $(D_T V = D_V T)$. □

当 $v=0$ 时. $\gamma_u(t)=\exp_{\gamma(0)} t(\dot\gamma(0)+uw)$

$$U(t)=\frac{\partial}{\partial u}\gamma_u(t)\Big|_{u=0} = d\exp_{\gamma(0)}\Big|_{t\dot\gamma(0)}(tw)$$



$\Downarrow$

$\gamma(t)\big|_{[0,l]}$ ┆ $\gamma(t_0)$ 沿 γ 与 $\gamma(0)$ 共轭 (i.e. $d\exp_{\gamma(0)}\big|_{t_0\dot\gamma(0)}$ 退化)

$\Leftrightarrow$ $\exists$ 沿 $\gamma(t)$ 的非零 Jacobi 场 $U(t)$

s.t. $U(0)=U(t_0)=0$

Cartan-Hadamard Thm

给定 (M,g) 完备 单连通 $\sec\le 0$ $p\in M$. 则

(1) $\exp_p$ 无退化点.

(2) $\exp_p$ 是微分同胚 (证是个覆盖映射).

证明: (1) 设 $\gamma(t)\big|_{[0,+\infty)}$ 是从 p 出发的测地线

$U(t)$ 是沿 γ 的非零 Jacobi 场.

则只需证 $U(t)\neq 0$. ($t>0$. const)

$\because T\langle U(t),U(t)\rangle = 2\langle U, D_T U\rangle$

$\dot\gamma(t)$

$$\therefore \langle U,U\rangle'' = \underbrace{2\langle D_T U, D_T U\rangle}_{\ge 0} + 2\langle U, \underbrace{D_T D_T U}_{U''}\rangle \ge 0$$

$D_T D_T U = -R(T,U)T$

$2\langle U, D_T D_T U\rangle = -2R(T,U,T,U) \ge 0$ $(\because \sec\le 0)$

起点为0. 又是凸的.

$\therefore$ $U(t)\neq 0$. 否则 全为0 才对!

作业：设 $\gamma(t_0)$ 沿 γ 不是 $\gamma(0)$ 的共轭点.

则 $\forall\, v\in T_{\gamma(0)}M,\ w\in T_{\gamma(t_0)}M$

$\exists!$ Jacobi 场 $U(t)$ s.t. $U(0)=v$, $U(t_0)=w$

$$\ddot{U}(t)+R(\dot\gamma,U)\dot\gamma=0$$

§4. 第一第二变分公式

一、第一……

C^∞ 的 $\gamma:[a,b]\times(-\varepsilon,\varepsilon)\longrightarrow(M,g)$

$$(t,u)\longmapsto \gamma_u(t)$$

$$L(u)=\int_a^b \dot\gamma_u(t)\,dt$$

$$L'(u)=\frac{d}{du}\int_a^b|\dot\gamma_u(t)|\,dt$$

$$=\int_a^b\frac{\partial}{\partial u}\sqrt{\langle T_u(t),T_u(t)\rangle}\,dt$$

$$=U(t)\sqrt{\langle T,T\rangle}=\frac{1}{|T|}\langle D_UT,T\rangle \quad (\text{connection 保内积})$$

$$=\frac{1}{|T|}\left(T\langle T,U\rangle-\langle D_TT,U\rangle\right)$$

$\therefore$ 若 $|\dot\gamma_0(t)|=1$，则

$$L'(0)=\int_a^b\left(T\langle T,U\rangle-\langle\ddot\gamma_0(t),U\rangle\right)dt$$

$$=\langle T,U\rangle\Big|_a^b-\int_a^b\langle\ddot\gamma_0(t),U(t)\rangle\,dt$$

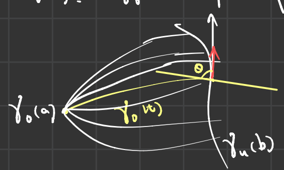
注① 若 $\gamma_u(a)=\gamma_0(a)$，$\gamma_u(b)=\gamma_0(b)$，即两端固定

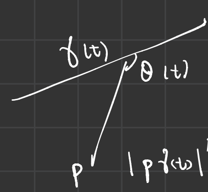
则 $L'(0)=-\int_a^b\langle\ddot\gamma_0,U\rangle\,dt$

（$\therefore$ 若 $\gamma_0(t)$ 是测地线，则关于固定端点的所有变分中 $L(0)$ 为极值）.

② 若 $\gamma_u(a)=\gamma_0(a)$，$\gamma_0(t)$ 为测地线.（$|\dot\gamma_0(t)|=1$）

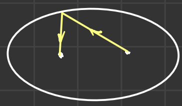
$|U_0(b)|=1$





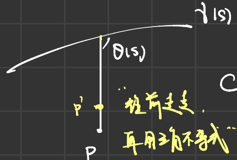
$|p\gamma(t)|'=-\cos\theta(t)$

则 $L'(0) = -\cos\theta$

作业: 

利用第一变分式

证明: 光会过焦点 (如右)

③ 

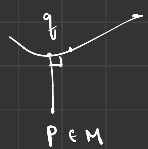
$C(M,g)$

$d'(p,\gamma(s)) \le -\cos\theta(s)$　证明: 进行变分 (光滑).

(导数存在性?)

且 "=" $\Longleftrightarrow$ $[p\gamma(s)]$ 与 $\gamma(s)$ 夹最小角 θ.

∵ p 直接过去不一定是光滑变分, 两点之间可能存在多条测地线.

推论: 

$N \hookrightarrow M$

$d(p,q)$ 为 $d(p,q')\big|_{q'}$ 的极小值

$\Longrightarrow [p,q] \perp N$.

二. "第二变分式"

$$L''(u) = \frac{d^2}{du^2}\int_a^b |\gamma_u'(t)|\,dt$$

$$= U\left(\frac{1}{|T|}\langle D_U T, T\rangle\right) = U\left(\frac{1}{|T|}\langle T, D_T U\rangle\right)$$

$$= -\frac{1}{|T|^2}\left(\frac{1}{|T|}\langle D_T U, T\rangle\right)\langle D_T U, T\rangle + \frac{1}{|T|}\langle D_T U, D_T U\rangle$$

$$+ \frac{1}{|T|}\langle T, D_U D_T U\rangle$$

$$= -\frac{1}{|T|^3}\langle D_T U, T\rangle^2 + \frac{1}{|T|}\langle D_T U, D_T U\rangle + \frac{1}{|T|}\underbrace{\langle T, D_U D_T U\rangle}_{\substack{-R(U,T,U,T)\\ +\langle T, D_T D_U U\rangle}}.$$

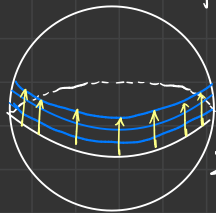
∵ 若 $\gamma_0(t)$ 为测地线且 $\overset{\substack{|T|\\ \parallel}}{|\dot{\gamma}_0(t)|} = 1$

则上式 $L''(0) = \int_a^b \Big\{ \underbrace{-\langle T, D_T U\rangle^2 + |\dot{U}(t)|^2}_{} - R(U,T,U,T) + \langle T, D_T D_U U\rangle \Big\}\,dt$

$$= \langle T, D_U U\rangle\Big|_a^b + \int_a^b \Big[\,|\dot{U}^{\perp}(t)|^2 - R(T,U^{\perp},T,U^{\perp})\Big]\,dt$$

∵ $D_T T = 0$.

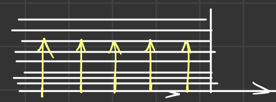
例：　　异非测地变分，　U是单位的



单位球

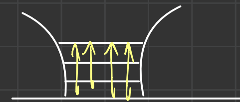
$L''(0) = -\int_a^b dt < 0$

U



T

$\mathbb{R}^2$.　$L''(0) = 0$



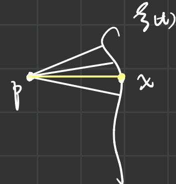
$\mathbb{H}^2(-1)$

$L''(0) = \int_a^b dt > 0$

推论. (M, g) 是 Hadamard mnfd.

$\Rightarrow$ $D^2\rho^2$ 正定. 其中 $\rho(x) = d(p, x)$. $p, x \in M$

证明: $D^2\rho^2(\dot\gamma(t), \dot\gamma(t))\Big|_{t=0} \geq 0$　　"$\gamma(0) = \gamma_x$"



两点间测地线唯一.

$x \neq p$　$= \dot\gamma\dot\gamma(\rho^2)\Big|_{t=0} - \underbrace{D_{\dot\gamma}\dot\gamma(\rho^2)}_{=0\ (\text{可设 } \gamma \text{ 为测地线})}\Big|_{t=0}$

$= \frac{d^2\rho^2}{dt^2}\Big|_{t=0}$

$= 2(\rho\cdot\dot\rho)' = 2(\dot\rho)^2 + 2\rho\cdot\ddot\rho\Big|_{t=0}$　$\geq \int_a^b |U_{(t)}^{\perp}|^2 dt > 0$

∵ 截面曲率 ≤ 0. $\Rightarrow -R \cdots \geq 0$.

若γ的切方向与 $[p, x]$ 重合，则 $= 2(\dot\rho)^2 = 2$.

∴ 一定大于 > 0

若 p 与 x 重合呢？　原点处 $D^2\rho^2 > 0$ ✓.

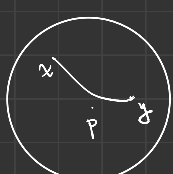
□

ρ^2 限制在流形上是凸函数

进而 此流形上，任一测地球是凸的.

即连接 x, y 的测地线仍在球中.

（最大值在端点、$d(x,p)$ or $d(y,p) \geq d(M,p)$　M在 a, b 间）



"正曲率不太行，双曲可以."
如球面

测地凸邻域存在性？

推论：（Synge定理） 1936

(M^{2m}, g) 完备,紧（closed），$\sec > 0$，可定向 $\Longrightarrow$ M 单连通

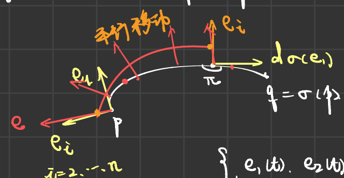
Weinstein Thm：

$1-1$，+ 微分同胚 ...

条件如 Synge 定理，且 $\sigma: M \longrightarrow M$ 是一个等距，保定向的

则 σ 一定有不动点。

证明：可设 $d(p, \sigma(p)) = \min d(x, \sigma(x))$ （compact）



$\{e_1(t), e_2(t), \dots, e_n(t)\}$

$d\sigma^{-1} \circ P_\gamma$ 保定向、等距、奇数维 $\Longrightarrow$ "有特征值为1" 即一定有一个与 e_1 垂直的向量 ξ。

$$L(u) = L\left(\exp_{\gamma(t)}(u \cdot \xi(t))\right)$$

$$L''(0) = \underbrace{\langle T, D_U U\rangle \Big|_a^b}_{0} + \int_a^b \Big(\underbrace{|U^{\perp\prime}(t)|^2}_{0} - R(T, U^\perp, T, U^\perp) \Big) dt$$

$$= -\int_a^b R \, dt < 0$$

即红线在更短，与 p 选取矛盾 $\Longrightarrow$ 有不动点。 $\square$

「如果有基本群，有覆盖，不存在不动点。

但现在有，$\therefore$ 无基本群。」

给偶数维分类：两种。

奇数维？

$\mathbb{RP}^2$ 不可定向。$\mathbb{RP}^3$ 可定向 ...

$\forall X(t), Y(t)$ （设 $\gamma(t)|_{[a,b]}$ 测地线）

定义：$I(X, Y) = \int_a^b \left(\langle \dot X, \dot Y \rangle - R(T, X, T, Y) \right) dt$

[CE] Cheeger-Ebin.

$$I(X,X)=\int_a^b\left[\,|\dot X|^2-R(T,X,T,X)\right]dt$$

$$I(X,X)=\int_a^b\left[\,|\dot X|^2-R(\dot\gamma,X,\dot\gamma,X)\right]dt$$

注：当 $X(t)\perp\dot\gamma(t)$，$I(X,X)$是 $\quad L''\left(\exp_{\gamma(t)} uX(t)\Big|_{[a,b]}\right)\Big|_{u=0}=I(X,X)$

指标引理：测地线 $\gamma(t)\big|_{[0,1]}\subset(M,g)$，$\gamma(t)$都不是（沿$\gamma$）$\gamma(0)$的共轭点。

$\forall\, W(t)\big|_{[0,1]}$（沿$\gamma(t)$），其中 $W(0)=0$，$\exists!$ Jacobi 场 $V(t)$ s.t. $V(0)=W(0)=0$，$V(1)=W(1)$

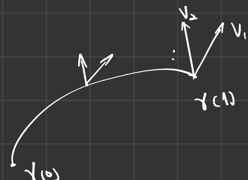
而且 $I(V,V)\leq I(W,W)$

作业：$W(0)=0$ ..., $I(0)=W(0)$. case.

若 X为Jacobi场，$I(X,X)=\int_a^b(\langle\dot X,X\rangle)'\,dt=\langle\dot X,X\rangle\big|_a^b$

$-R(\dot\gamma,X,\dot\gamma,X)=\langle\ddot X,X\rangle$

$\ddot X+R(\dot\gamma,X)\dot\gamma=0$



证明：设 $V_1\cdots V_n$ 为 $T_{\gamma(1)}M$ 处一组基

则 $\exists!$ 一组 Jacobi 场，$V_i(t)\big|_{[0,1]}$

s.t. $V_i(0)=0$，$V_i(1)=V_i$

而且$\{V_i(t)\}_{t>0}$，<u>处处线性无关</u>(?) $\quad\dot V_i(0)=A_i\neq0$.

进而可设 $V_i(t)=tA_i(t)$，其中 $A_i(0)=A_i$

注意可令 $W(t)=\sum_{i=1}^n f_i(t)V_i(t)$，其中 $f_i(t)\in C^\infty([0,1])$.

$\quad\| \qquad \sum g_i(t)A_i(t)\,t \qquad \|$

$\qquad\quad \cancel{t}$

$$V(t)=\sum_{i=1}^n f_i(1)V_i(t)$$

$$I(V,V)=\langle\dot V,V\rangle\Big|_0^1=f_i(1)f_j(1)\langle\dot V_i(1),V_i(1)\rangle$$

$$I(W,W)=\int_0^1\left[\langle\dot W(t),\dot W(t)\rangle-R(\dot\gamma(t),W(t),\dot\gamma(t),W(t))\right]dt$$

$\dot W(t)=\sum_{i=1}^n\left[\dot f_i(t)V_i(t)+f_i(t)\dot V_i(t)\right]=A+B$

$$=\int_0^1\left(\langle A,A\rangle+2\langle A,B\rangle+\langle B,B\rangle\right)dt-\int_0^1R\,dt$$

其中 $\int_0^1\langle B,B\rangle dt=\int_0^1\langle\sum f_i(t)\dot V_i(t),\sum f_i(t)\dot V_i(t)\rangle dt$

$$=\int_0^1\sum_i\sum_j f_i(t)f_j(t)\langle\dot V_i(t),\dot V_j(t)\rangle dt$$

$$\int_0^1 \langle B,B\rangle dt = \int_0^1 \sum_i\sum_j f_if_j\left(\langle \dot V_i, V_i\rangle' - \langle \ddot V_i(t), V_j(t)\rangle\right) dt$$

integral by parts

$R(\dot\gamma, V_i, \dot\gamma, V_j)$ 换为 $R(\dot\gamma, W, \dot\gamma, W)$

$$= \sum_i\sum_j f_if_j \langle \dot V_i, V_i\rangle\Big|_0^1 - \sum_i\sum_j (f_if_j)' \langle \dot V_i, V_j\rangle dt$$

$I(V,V)$

$(f_if_j)' \to f_i'f_j + f_if_j'$

$-2\langle A, B\rangle$

断言 $\langle \dot V_i, V_j\rangle = \langle V_i, \dot V_j\rangle$

$$(左-右)' = \left(\langle \dot V_i, V_j\rangle - \langle V_i, \dot V_j\rangle\right)'$$

$$= \langle \ddot V_i, V_j\rangle + \langle \dot V_i, \dot V_j\rangle - \langle \dot V_i, \dot V_j\rangle - \langle V_i, \ddot V_j\rangle$$

$$= \langle \ddot V_i, V_j\rangle - \langle V_i, \ddot V_j\rangle$$

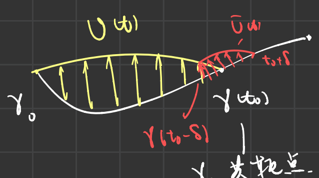
$$= R(\dot\gamma, V_i, \dot\gamma, V_j) - R(\dot\gamma, V_j, \dot\gamma, V_i) = 0$$

$\because$ initial value $= 0$

$\therefore$ " $\equiv 0$ "

$\square$

推论



$\Rightarrow$ $\gamma(t)\big|_{[0,1]}$ 不是最短线

$$W = \begin{cases} U(t) & t\in[0,t_0] \\ 0 & t\in[t_0,1] \end{cases}, \quad I(W,W) = 0$$

$$V(t) = \begin{cases} U & t\in[0,t_0-\delta] \\ \bar U & t\in[t_0-\delta, t_0+\delta] \\ 0 & t\in[t_0+\delta, 1] \end{cases}$$

$I(V,V) \le I(W,W) = 0$

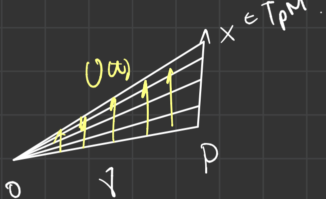
其中 $f(a)=0$, $f(b)=0$. ✓

$\Rightarrow$ 当 $b>\pi$ 时 $\exists\ C^\infty\ f(t)\big|_{[a,b]}$ s.t. $\int_a^b (\dot f^2 - f^2)\,dt < 0$.

"单位球面上"

(Bonnet – Myers thm)

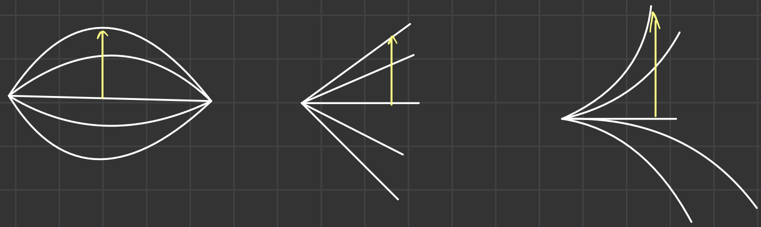
作业: (M,g) 完备. $Ric_M \ge n-1 \Rightarrow diam(M) \le \pi$



$\dot U(0) = \frac{X}{r}$

$d\exp\big|_p (X)$

$$U(t) = \frac{\partial}{\partial u}\left[\exp_p\left(\frac{t}{r}(p+uX)\right)\right]$$



越来越大

Rauch 比较定理　$M, \tilde{M}, \gamma(t)\big|_{[0,1]}, \tilde{\gamma}(t)$

$K(\pi) \geq K(\tilde{\pi})$　π 是含 $\dot{\gamma}(t)$ 的任一平面

$\tilde{\pi}$ ... $\dot{\tilde{\gamma}}(t)$ ......

截面曲率？

$\gamma(t)\big|_{[0,1]}$ 均不是 $\gamma(0)$ 的共轭点

$V(t), \tilde{V}(t)$ 为沿 $\gamma, \tilde{\gamma}$ 的 Jacobi 场

且垂直于

而且 $V(0)=\tilde{V}(0)=0$, $|\dot{V}(0)| = |\dot{\tilde{V}}(0)|$, $\langle \dot{V}(0), \dot{\gamma}(0)\rangle = \langle \dot{\tilde{V}}(0), \dot{\tilde{\gamma}}(0)\rangle$

$\Longrightarrow$　$|V(1)| \leq |\tilde{V}(1)|$

Proof: $\because \lim_{t\to 0} \dfrac{\langle V, V\rangle}{\langle \tilde{V}, \tilde{V}\rangle} \overset{?}{=} \lim_{t\to 0} \dfrac{2\langle \dot{V}, V\rangle}{2\langle \dot{\tilde{V}}, \tilde{V}\rangle} \overset{?}{=} \lim_{t\to 0} \dfrac{2\langle \dot{V}, \dot{V}\rangle}{2\langle \dot{\tilde{V}}, \dot{\tilde{V}}\rangle} = 1$

$\therefore$ 只需证 $\left(\dfrac{\langle V, V\rangle}{\langle \tilde{V}, \tilde{V}\rangle}\right)' \leq 0$ 即可

$\Longleftarrow$　$2\langle \dot{\tilde{V}}, \tilde{V}\rangle \langle V, V\rangle - 2\langle \tilde{V}, \tilde{V}\rangle \langle \dot{V}, V\rangle \geq 0$

$\Longleftarrow$　$\dfrac{\langle \dot{\tilde{V}}, \tilde{V}\rangle}{\langle \tilde{V}, \tilde{V}\rangle}\Big|_{t_1} \geq \dfrac{\langle \dot{V}, V\rangle}{\langle V, V\rangle}\Big|_{t_1}$　$\forall\, t_1 > 0$.

可能为0。　V 不为0，无共轭点.

令 $\tilde{W}(t) = \dfrac{\tilde{V}(t)}{\tilde{V}(t_1)}$　　$W(t) = \dfrac{V(t)}{|V(t_1)|}$

$\therefore$ 只需证 $\langle \dot{\tilde{W}}(t_1), \tilde{W}(t_1)\rangle \geq \langle \dot{W}(t_1), W(t_1)\rangle$

$\| \; I_{[0,t_1]}(\tilde{W}, \tilde{W})$　　　$\| \; I_{[0,t_1]}(W, W)$

两个流形上，不能用指标引理.

令 $\bar{W}(t) = |\tilde{W}(t)|\, e(t)$. 其中 $e(t)$ 沿 γ 平行 且 $|e(t)|=1$, $e(t_1) = W(t_1)$

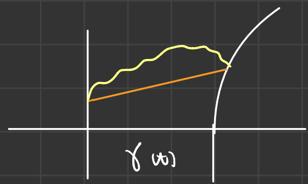
$\therefore I(W, W) \underset{\text{指标引理}}{\leq} I(\bar{W}, \bar{W}) = \int_0^1 \left[|\dot{\bar{W}}|^2 - R(\dot{\gamma}, \bar{W}, \dot{\gamma}, \bar{W})\right] dt$

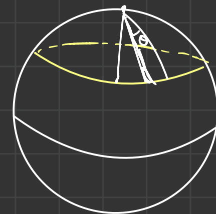
$\dot{\bar{W}} = (|\tilde{W}(t)|)' e(t)$ [$\because e(t)$ 沿 γ 平行 $\therefore \dot{e}(t) = 0$]　$\leq \int_0^1 \left[|\dot{\tilde{W}}|^2 - R(\dot{\gamma}, \tilde{W}, \dot{\gamma}, \tilde{W})\right] dt = I(\tilde{W}, \tilde{W})$.

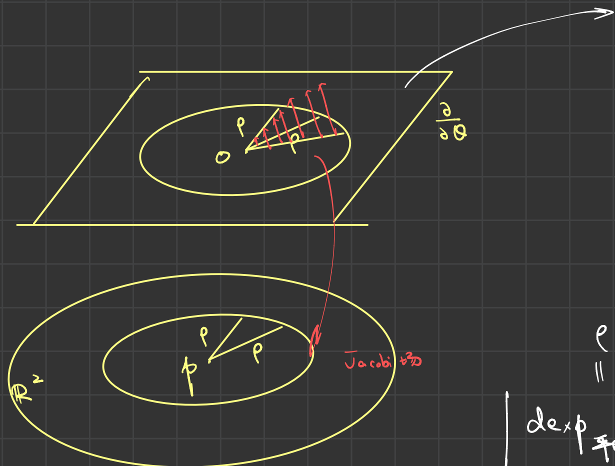
$$|\dot{W}|^2 = \left(|\tilde{W}(t)|\right)'^2 \le |\dot{\tilde{W}}|^2$$

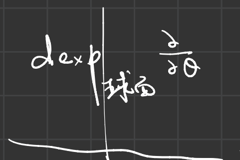
$\tilde{W}(t) = |\tilde{W}(t)|\,\tilde{e}(t)$ 不一定平行

$\therefore (\tilde{W}(t))' = |\tilde{W}(t)|'\,\tilde{e}(t) + |\tilde{W}(t)|\,\tilde{e}'(t)$









$$\frac{\partial}{\partial\rho}\left(d\exp \frac{\partial}{\partial\theta}\right)\Big|_{\rho=0} = \frac{\partial}{\partial\theta}$$

ρ "Rauch 比较定理" $\sin\rho$

$\|$ $\|$

$$\left|d\exp_{平面} \frac{\partial}{\partial\theta}\right| > \left|d\exp_{球面} \frac{\partial}{\partial\theta}\right|$$

$$K_{平} = 0 < 1 = K_{球}$$

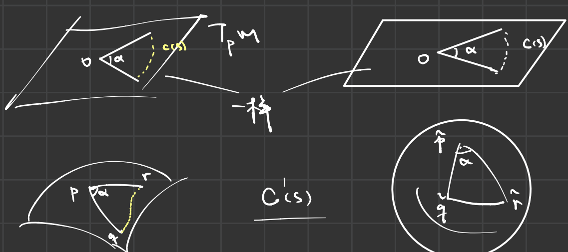
Toponogov 定理 (局部)

(M,g), $\sec \ge k$. $[p,q], [p,r] \subset B_p(\rho)$, 其中 $\exp$ 在 $B_0(\rho)$ 上是微分同胚

在 S_k^2 上取相应的 $[\tilde{p}\tilde{q}]\,[\tilde{p}\tilde{r}]$

若 $\angle qpr = \angle \tilde{q}\tilde{p}\tilde{r}$ 则 $|qr| \le |\tilde{q}\tilde{r}|$

证明:

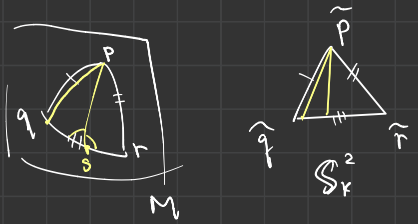


记 $\frac{\partial}{\partial s}$, $\widetilde{\frac{\partial}{\partial s}}$ 分别为沿 ρ 线的 Jacobi 场 → $\exp_p(\rho c(s))$

而 $\left(\frac{\partial}{\partial s}\right)'\rho\Big|_{\rho=0} = \left(\widetilde{\frac{\partial}{\partial s}}\right)'\rho\Big|_{\rho=0} = c'(s)$

∴ 由 Rauch $\left|\frac{\partial}{\partial s}\right| \le \left|\widetilde{\frac{\partial}{\partial s}}\right|$ ✓

等价版本:



$\Rightarrow \angle qpr \ge \angle \tilde q \tilde p \tilde r$

整体版本

Toponogov Thm'

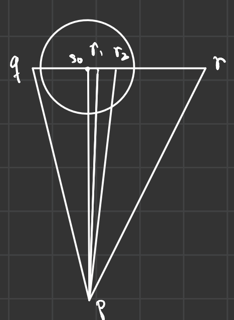
(M,g) 完备 $\sec \ge k$ $[p,q]$ $[p,r] \subset M$，在 S_k^2 上取

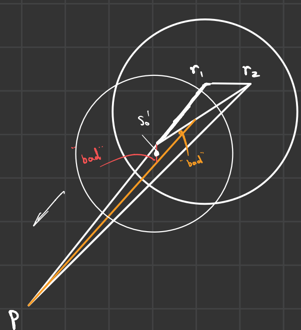
相应的 $[\tilde p, \tilde q]$ $[\tilde p, \tilde r]$，若 $\angle qpr = \angle \tilde q \tilde p \tilde r$，则 $|qr| \le |\tilde q \tilde r|$

Lemma. 设 $\angle pqr$ 坏角 则在 $[qr]^\circ$ 上存在一点 s，s.t. $\angle psq$ 或 $\angle psr$ 坏.

($k=0$ 时) 且 $|ps| \le \max\{|pq|, |pr|\}$ 特别地，当 $|qs| \le |qp|$ 时，$\angle psq$ 坏

推论 设 $\angle pqr$ 坏，则在 $[qr]$ 上存在点 s_0 s.t. $\forall \delta > 0$，$\exists r_1, r_2 \in B_{s_0}(\delta_0) \cap [qr]$ s.t. $\angle p r_1 r_2$ 坏



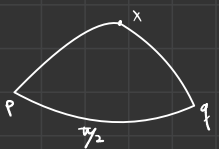


一. $\sec_M \geq 1$

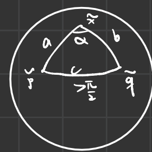
定理: $\mathrm{diam} \leq \pi$; "=" $\Longleftrightarrow M \overset{\text{iso}}{\simeq} S^n(1)$ (Toponogov)

定理: 若 $\mathrm{diam} > \pi/2$, 则 $M \overset{\text{home}}{\simeq} S^n$

(Grove-Shiohama)
1977



Topo $\Longrightarrow$



$\alpha > \pi/2$

$\cos c = \cos a \cos b + \sin a \sin b \cos\theta$

e.g. $\begin{cases} 1 \leq \sec \leq 4 \\ \mathrm{diam} = \pi/2 \end{cases}$

$S^n(\frac{1}{2})$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, C^2

∵不单连通

定理 (Berger) $\mathrm{diam} = \pi/2$, $1 \leq \sec \leq 4$ + 单连通 $\Longrightarrow M \overset{\text{iso (等距)}}{\simeq} S^n(\frac{1}{2})$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, C^2

Cayley平面

球面定理 (Klingenberg) $1 < \sec \leq 4 \Longrightarrow M \overset{\text{home}}{\simeq} S^n$

diff 2009

Thm Gromoll-Meyer 1969

$\left.\begin{array}{l} \sec_M > 0 \\ \text{非紧} \end{array}\right\} \Longrightarrow M \overset{\text{diff}}{\simeq} \mathbb{R}^n$

soul is a point.

Soul Thm 1970 Cheeger-Gromoll

$\sec_M \geq 0$ 非紧 $\Longrightarrow \exists$ Soul S

紧全测地子流形

s.t. $M \overset{\text{diff}}{\simeq} N(S)$

Soul 猜想 Conj. $\sec_M$ 在一点处正 $\Longrightarrow$ Soul 是一个点

1994 Perelman is true.

三. Gromov's 几乎平坦流形

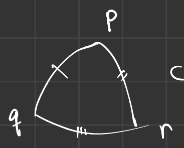
(M,g) 紧

$\exists\, \varepsilon(n) > 0$, s.t. $\sec \cdot \mathrm{diam}^2 < \varepsilon(n)$

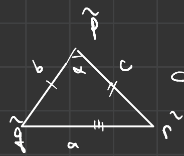
$\Rightarrow \tilde{M}$ (万有覆盖) 是一个幂零群.

Gromov-Hausdorff 收敛理论.

Toponogov. (M,g) $\sec_M \geq k$. 完备



$\subset M$



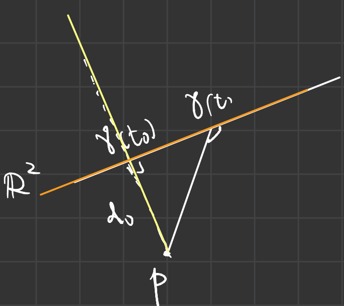
$\subset S_k^2$

球面上余弦定理

$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha$

$\Downarrow$

$\angle qpr \geq \angle \tilde{q}\tilde{p}\tilde{r}$



$$\left(|p\gamma(t)|^2\right)'' = 2$$

$$\| \quad d_0^2 + (t-t_0)^2$$

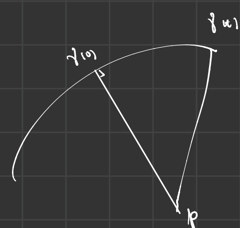
$$|p\gamma(t)|'' = \left(\frac{1}{\sqrt{d_0^2+(t-t_0)^2}}(t-t_0)\right)' = \frac{1}{\sqrt{d_0^2+(t-t_0)^2}} - \frac{(t-t_0)^2}{\left(d_0^2+(t-t_0)^2\right)^{3/2}}$$

$$\therefore |p\gamma(t)|''\Big|_{t=t_0} = \frac{1}{d_0}$$

考虑 $d(p,x)$, $x \in \mathbb{R}^2 \setminus \{p\}$

$\Delta d = \frac{1}{d}$ $(d \neq 0)$

$S^2(1)$



$$\left(1-\cos|p\gamma(t)|\right)'' = \left(1-\cos d_0 \cos t\right)''$$

$$= \cos d_0 \cos t$$

$$\Rightarrow \left(1-\cos|p\gamma(t)|\right)'' + 1-\cos|p\gamma(t)| = 1$$

$$f''(t) + k f(t) = 1$$

$$f(t) = \begin{cases} 1-\cos|p\gamma(t)| & k=1 \\ \frac{1}{2}|p\gamma(t)|^2 & k=0 \\ \cosh|p\gamma(t)|-1 & k=-1 \end{cases}$$

$$\left(\cos|p\gamma(t)|\right)'' = \left(\cos d_0 \cos t\right)'' = -\cos d_0 \cos t$$

$\|$

$$\left(-\sin|p\gamma(t)|\cdot(|p\gamma(t)|)'\right)' = -\cos|p\gamma(t)|\left[(|p\gamma(t)|)'\right]^2 - \sin|p\gamma(t)|\cdot|p\gamma(t)|''$$

$-\cos\theta = 0$

$\because$ 第一变分 + $\theta = \pi/2$

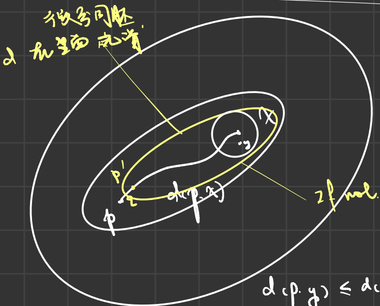
$\because \left(|\dot{\varphi} \gamma(t)| \right)'' \Big|_{t=0} = \dfrac{\cos d_0}{\sin d_0}$ Hessian 特征值之一. 另一个为 0.

$\therefore \Delta d = \dfrac{\cos d}{\sin d}$

$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$. （双曲空间 $K=-1$. 时）.

$$\Delta d = \frac{\cosh d}{\sinh d}$$

一. $Ric_M \geq (n-1)k$. (M^n, g) 完备



Thm: $\Delta d \leq \begin{cases} \cot d & k=1 \\ \frac{1}{d} & k=0 \\ \coth d & k=-1 \end{cases}$

$d(p,y) \leq \underline{d(p',y)} + \varepsilon$

smooth

$y = x \Rightarrow$ "="

$\Delta d(p',x) \leq \dfrac{1}{d(p',x)}$

$f \in C[a,b]$, $f''(t_0) \leq B$ （在 support 意义下） 多元时 Δ

$t_0 \in [a,b]$

$\Updownarrow$

$\exists A$, s.t. $f(t_0+\tau) \leq f(t_0) + A\tau + \frac{1}{2}B\tau^2 + o(\tau^2)$

$\Updownarrow$

$\exists f_\varepsilon(t) \in C^2(t_0-\delta, t_0+\delta)$ s.t. $\begin{cases} f_\varepsilon(t_0) = f(t_0) \\ f_\varepsilon(t_0+\tau) \geq f(t+\tau) \\ f_\varepsilon''(t_0) = B+\varepsilon \end{cases}$

定理: $\dfrac{Vol(B_p(r))}{Vol(B_{\tilde{p}}(r))} \downarrow r \uparrow$ 且 $\lim\limits_{r\to 0} \dfrac{Vol(B_p(r))}{Vol(B_{\tilde{p}}(r))} = 1$.

$\left(\Rightarrow Vol(B_p(r)) \leq Vol(B_{\tilde{p}}(r)) \right.$,且当 "=" $\left. \Leftrightarrow B_p(r) \overset{iso}{\simeq} B_{\tilde{p}}(r) \right)$.

"$n=2$"

$$g = \begin{cases} d\rho^2 + \sin^2\theta\, d\theta^2 & k=1 \\ d\rho^2 + \rho^2 d\theta^2 & k=0 \\ d\rho^2 + \sinh^2\rho\, d\theta^2 & k=-1 \end{cases}$$

$$Vol(B_p(r)) = \begin{cases} \int_0^{2\pi}\int_0^r \sin\rho\, d\rho\, d\theta = (1-\cos r)\,2\pi \\ \iint \rho\, d\rho\, d\theta = \pi r^2 \end{cases}$$

$\sin\rho < \rho \Longrightarrow$ 体积比较.

$\frac{\sin\rho}{\rho}\downarrow \Longrightarrow$ 相对体积比较

$$g = d\rho^2 + (d\theta^1, \cdots, d\theta^{n-1}) \underbrace{(g_{ij})_{(n-1)\times(n-1)}}_{G} \begin{pmatrix} d\theta^1 \\ \vdots \\ d\theta^{n-1} \end{pmatrix}$$

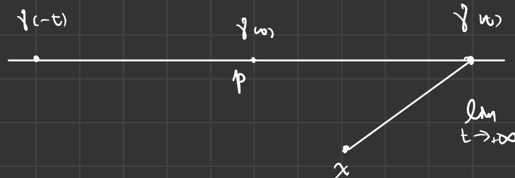
$$\mathrm{Vol}(B_p(r)) = \int_{S^{n-1}(1)} \int_0^r \sqrt{|G|}\, d\rho\, d\theta \quad , \quad |G| = \left| \frac{\partial}{\partial\theta^1} \wedge \cdots \wedge \frac{\partial}{\partial\theta^{n-1}} \right|$$

$$sn_k^{n-1} r \longleftarrow \frac{|G|}{\left|\frac{\tilde\partial}{\partial\theta^1} \wedge \cdots \wedge \frac{\tilde\partial}{\partial\theta^{n-1}}\right|} \downarrow \quad r\uparrow$$

(1971)

Thm. $k=0$ 时. 若 M 含有一条直线 $\Rightarrow M \overset{iso}{\cong} N\times\mathbb{R}$

proof:



$$\lim_{t\to+\infty} d(x,\gamma(t)) - t \triangleq b(x)$$

$$\Delta(b^+ + b^-) \le 0 \Leftarrow \{\Delta b^+ \le 0,\quad \Delta b^- \le 0\}$$

二. $Ric \ge (n-1)$

Thm $\mathrm{diam} \le \pi$ 取 "=" $\Leftrightarrow M \overset{iso}{\cong} S^n(1)$

三 $Ric \ge 0$

Thm: (M,g) 完备紧 $\Rightarrow b_1(M) \le \dim(M)$ "=" $\Leftrightarrow M \overset{iso}{\cong} S^n(1)$

1990 thm 非紧 $\Rightarrow b_1(M) \le \dim(M)-1$ "=" ?

22年 叶锷. "=" $\Leftrightarrow$ M平坦

四. $Ric \geq -(n-1)$

定理(紧致): 体积熵 (entropy) $h(M) \leq n-1$, $h(M) = \lim_{r\to\infty} \dfrac{\log(Vol(B_p(r)))}{r}$

"=" ⟺ 且 M 紧 时 M 万有覆盖: $\tilde{M} \overset{iso}{\simeq} H^n(-1)$

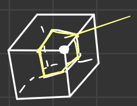
CBB & CAT Space
(CBA)

Toponogov $sec_M \geq k \Rightarrow \angle pqr \geq \angle_k \tilde{p}\tilde{q}\tilde{r}$

当 $k=0$ 时

$\Leftrightarrow \ |pr|^2 = |pq|^2 + |qr|^2 + 2|pq||qr| \cos \tilde{\angle} pqr$

$|pr|^2 \leq |pq|^2 + \cdots\cdots 2|pq||qr| \cos \angle pqr.$



NOT Riemann manifold.

∵ ∃ "尖点"

但仍有

一. Def (X, d) 内蕴度量空间

$\Leftrightarrow \ \forall p, q \in X, \ \forall \delta > 0, \ \exists z_1, \cdots, z_{n_\delta}$

p ⋯⋯ q

s.t. $|z_i z_{i+1}| < \delta, \ \left| \sum_{i=0}^{n_\delta - 1} |z_i z_{i+1}| - |pq| \right| < \delta.$

Def. 若 $\forall x \in X, \exists U_x$ s.t. $\forall a, b, c, d \in U_x$

b. .c 有 $\tilde{\angle}_k bac + \tilde{\angle}_k bad + \tilde{\angle}_k cad \leq 2\pi$

a. 则称 (X, d) 是 $cur \geq k$ 的空间

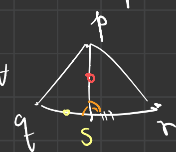
.d

再假设 (X,d) 是测地空间

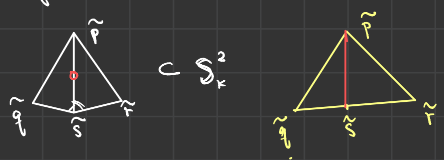
Huabobo.

考察图上赋曲率
研究优化

$[\mathrm{cur} \geq k]$ 等价条件：

$\forall$  $\subset U_x$ 有 $\tilde{\angle}_k psq + \tilde{\angle}_k psr \leq \pi$.

$(\angle qsp + \angle qsr + \angle psr \leq 2\pi$?$)$



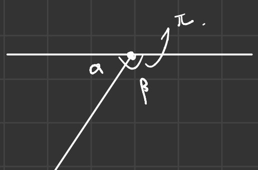
第一"等价条件" $|ps| \geq |\tilde{p}\tilde{s}|$

$\Downarrow$

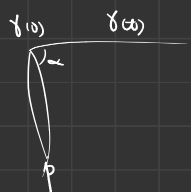
$\tilde{\angle}_k pqs \nearrow \quad s \to q \implies \angle pqr \geq \tilde{\angle}_k pqr$

推论：可定义角度 $\angle pqr = \lim\limits_{\substack{s\to q,\ s\in[q,r] \\ t\to q,\ t\in[q,p]}} \tilde{\angle}_k tqs$

Toponogy 比较定理

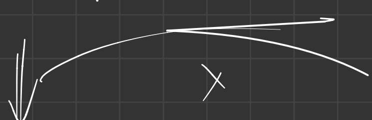
性质：①  $\Rightarrow \alpha+\beta=\pi$

② 第一变分公式成立



$|p\gamma(t)|'\big|_{t=0} = -\cos\alpha$

性质：测地线不分叉



可定义 $\Sigma_p X$（方向空间） mnfd's $\Sigma_p X = S^{n-1}$?

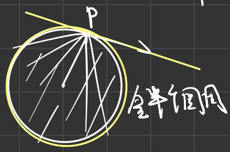
当 X 局部紧（且完备）

可定义 $\dim(X)$ 而且 $\Sigma_p X$ 是 curvature ≥ 1 的 Alexandrov space.

且 $\dim(\Sigma_p X) = \dim X - 1$

进而 X 上可如下定义边界点

若 $\Sigma_p X$ 含一边界点，则称 p 为一个边界点

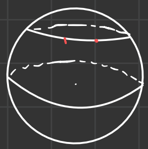


∴ p 是边界点

定理：X cpt, $\dim(X) < +\infty$. $\partial X \neq \phi$.

(Peleman)

$\Rightarrow$ D(X) 是 $cur \geq k$ 的空间

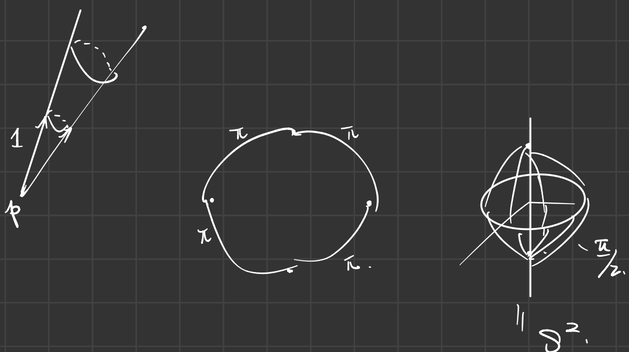


Conj: ∂X 是 $cur \geq k$ 的空间 (open)

?

例：① (M.g) $sec_M \geq k$

② $(M_i.g)$ $sec_{M_i} \geq k$. 且 $M_i \longrightarrow X$. 则 $Cur(X) \geq k$

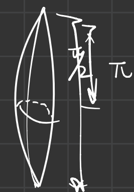


$T_p X$ 切锥 $cur(X) \geq 0$

$S^n * S^m = S^{n+m+1}$

↑

Join?



Thm X cpt. $CurX \geq 1$. $\Rightarrow diam \leq \pi$

(拓扑比较) Thm X cpt $cur(X) \geq k$.

$\forall a.b.c.d \in X$

$\Rightarrow \tilde{\angle}_k bac + \tilde{\angle}_k cad + \tilde{\angle}_k bad \leq 2\pi$

$OBA \Longleftrightarrow |ps| \le |\hat{p}\hat{s}|$

$\Updownarrow$

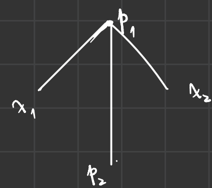
$\tilde{\angle}_k pqs \quad \downarrow s \to q$

定义为何空间曲率

CAT(0)

CAT(k) 比较

$\forall$



若(1) $\tilde{\angle}_k x_1 p_1 x_2 \le \tilde{\angle}_k x_1 p_1 p_2 + \tilde{\angle}_k x_2 p_1 p_2$

或(2) $\tilde{\angle}_k x_1 p_2 p_1 \le \tilde{\angle}_k x_1 p_2 p_1 + \tilde{\angle}_k x_2 p_1 p_2$，

或 上面的比较角有一个不存在，

则称 CAT(k) 比较。